2 Metric geometry

At this level there are two fundamental approaches to the type of geometry we are studying. The first, called the synthetic approach, involves deciding what are the important properties of the concepts you wish to study and then defining these concepts axiomatically by their properties. This approach was used by Euclid in his Elements (around 300 B.C.E.) and was made complete and precise by the German mathematician David Hilbert (1862-1943) in his book Grundlagen der Geometrie [1899; 8th Edition 1956; Second English Edition 1921].

The second approach, called the metric approach, is due to the American mathematician, George David Birkhoff (1884-1944) in his paper "A Set of Postulates for Plane Geometry Based on Scale and Protractor" [1932]. In this approach, the concept of distance (or a metric) and angle measurement is added to that of an incidence geometry to obtain basic ideas of betweenness, line segments, congruence, etc. Such an approach brings some analytic tools (for example, continuity) into the subject and allows us to use fewer axioms.

A third approach, championed by Felix Klein (1849-1925), has a very different flavour that of abstract algebra–and is more advanced because it uses group theory. Klein felt that geometry should be studied from the viewpoint of a group acting on a set. Concepts that are invariant under this action are the interesting geometric ideas. See Millman [1977] and Martin [1982].

(2.1) Definition (distance function)
A distance function on a set $\mathcal{S}$ is a function $d : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ such that for all $P, Q \in \mathcal{S}$
(i) $d(P,Q) \geq 0$; (ii) $d(P,Q) = 0$ if and only if $P = Q$; and (iii) $d(P,Q) = d(Q,P)$.

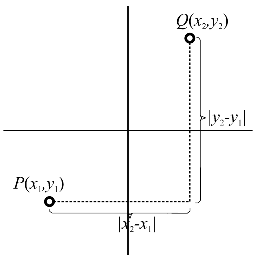
1. Let $\mathcal{M}$ denote non-empty set and let $d_M : \mathcal{M} \times \mathcal{M} \to \{0,1\}$ denote function on $\mathcal{M} \times \mathcal{M}$ defined on the following way: $\forall P, Q \in \mathcal{M}$

$$d_M(P,Q) = \begin{cases} 1, & \text{if } P \neq Q \\ 0, & \text{if } P = Q \end{cases}$$

Check is it d_M a distance function.

2. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ denote two points in $\mathbb{R}^2$, and let $d_{max} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ denote a function which is defined on the following way $d_{max}(P,Q) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$. Check is it d_{max} a distance function.

3. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ denote two points in $\mathbb{R}^2$, and let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ denote a function which is defined on the following way $d(P,Q) = \sqrt{(x_1 - x_2)^2 + 4(y_1 - y_2)^2}$. Check is it d a distance function.



(2.2) Definition (taxicab distance)
Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ denote two points in $\mathbb{R}^2$. The taxicab distance between P and Q is given by

$$d_T(P,Q) = |x_1 - x_2| + |y_1 - y_2|.$$

4. Show that the taxicab distance is a distance function on $\mathbb{R}^2$.

5. If d_0 and d_1 are distance functions on $\mathcal{S}$, prove that if $s \geq 0$ and $t > 0$, then $sd_0 + td_1$ is also a distance function on $\mathcal{S}$.

6. Let d denote distance function on $\mathcal{S}$, and define function $d' : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ on the following way: $\forall P, Q \in \mathcal{S}$

$$d'(P,Q) = \frac{d(P,Q)}{1 + d(P,Q)}.$$

Show that d' is also distance function on $\mathcal{S}$. Notice that $0 \leq d'(P,Q) < 1 \ \forall P, Q \in \mathcal{S}$.

(2.3) Definition (surjective function)
A function f from A to B is called onto, or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called a surjection if it is onto.

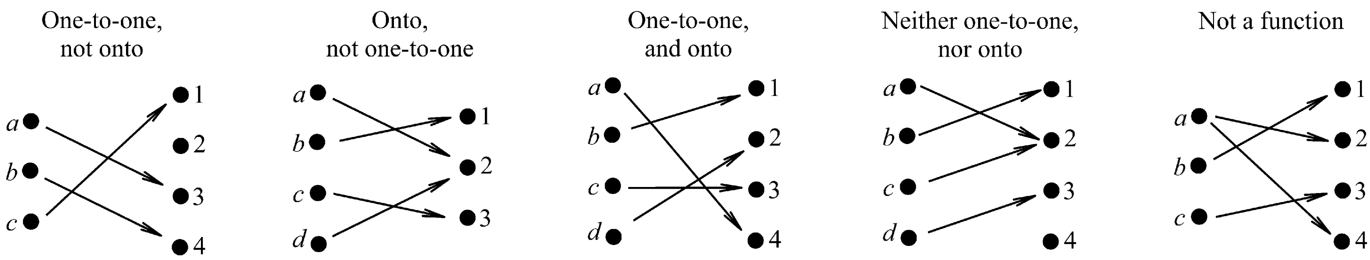
Remark: A function f is onto if $\forall y\ \exists x\ (f(x) = y)$, where the universe of discourse for x is the domain of the function and the universe of discourse for y is the codomain of the function.

7. Let $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$ denote Cartesian plane, and let $L_{m,n}$ denote non-vertical line. Lets define function $f : L_{m,b} \to \mathbb{R}$ on the following way; $f(P) = f((x,y)) = x\sqrt{1+m^2}$; where $P \in L_{m,b}$, $P(x,y)$. Show that f is surjection.

> **(2.4) Definition (injection)**
> A function f is said to be one-to-one, or injective, if and only if $f(x) = f(y)$ implies that $x = y$ for all x and y in the domain of f. A function is said to be an injection if it is one-to-one.

Remark: A function f is one-to-one if and only if $f(x) \neq f(y)$ whenever $x \neq y$. This way of expressing that f is one-to-one is obtained by taking the contrapositive of the implication in the definition. Note that we can express that f is one-to-one using quantifiers as $\forall x\ \forall y$ $(f(x) = f(y) \Rightarrow x = y)$ or equivalently $\forall x\ \forall y\ (x \neq y \Rightarrow f(x) \neq f(y))$, where the universe of discourse is the domain of the function.

> **(2.5) Definition (bijection)**
> The function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto.



Examples of Dirrerent Types of Correspondences

> **(2.6) Definition (inverse function)**
> Let f be a bijection from the set $\mathcal{A}$ to the set $\mathcal{B}$. The inverse function of f is the function that assigns to an element b belonging to $\mathcal{B}$ the unique element a in $\mathcal{A}$ such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

8. Let $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$ denote Poincaré plane, and let ${}_aL$ denote type I line. Lets define function $g : {}_aL \to \mathbb{R}$ on the following way $g(a,y) = \ln(y)$. Show that g is bijection, and determine inversion of g.

9. Let p denote line from incidence geometry $\{\mathcal{S}, \mathcal{L}\}$, and let $f : p \to \mathbb{R}$ denote surjection for which: $|f(P) - f(Q)| = d(P,Q)\ \forall P,Q \in p$; where d denote distance function on $\mathcal{S}$. Show that f is bijection.

> **(2.7) Definition ($\sinh(t)$, $\cosh(t)$, $\tanh(t)$, $\operatorname{sech}(t)$)**
> We define the hyperbolic sine, hyperbolic cosine, hyperbolic tangent and hyperbolic secant on the following way
> $$\sinh(t) = \frac{e^t - e^{-t}}{2}; \qquad \cosh(t) = \frac{e^t + e^{-t}}{2};$$
> $$\tanh(t) = \frac{\sinh(t)}{\cosh(t)} = \frac{e^t - e^{-t}}{e^t + e^{-t}}; \qquad \operatorname{sech}(t) = \frac{1}{\cosh(t)} = \frac{2}{e^t + e^{-t}}.$$

10. Show that for every value of $t \in \mathbb{R}$

(i) $[\cosh(t)]^2 - [\sinh(t)]^2 = 1$; (ii) $[\tanh(t)]^2 + [\operatorname{sech}(t)]^2 = 1$.

11. Let $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$ denote Poincaré plane, and let ${}_cL_r$ denote type II line. Lets define function $g : {}_cL_r \to \mathbb{R}$ on the following way: $f(x,y) = \ln(x - c + r) - \ln(y)$. Show that f is bijection, and determine inversion of f.

(2.8) Definition (Euclidean distance d_E)
Let $P(x_1, y_1)$ and $Q(x_1, y_1)$ denote two points in the Cartesian Plane $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$. The Euclidean distance d_E is given by

$$d_E = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

12. Let $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$ denote the Cartesian Plane and let $P(x_1, y_1)$ and $Q(x_1, y_1)$ denote two arbitrary points from Cartesian Plane. Show that Euclidean distance $d_E = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ is a distance function.

(2.9) Definition (Poincaré distance d_H)
Let $P(x_1, y_1)$ and $Q(x_1, y_1)$ denote two points in the Poincaré Plane $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$. The Poincaré distance d_H is given by

$$d_H = \begin{cases} |\ln(y_2) - \ln(y_1)|, & \text{if } x_1 = x_2 \\ |\ln(\frac{x_1 - c + r}{y_1}) - \ln(\frac{x_2 - c + r}{y_2})|, & \text{if } P, Q \in {}_cL_r \end{cases}.$$

13. Let $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$ denote the Poincaré Plane and let $P(x_1, y_1)$ and $Q(x_1, y_1)$ denote two arbitrary points from Poincaré Plane. Show that Poincaré distance d_H is a distance function.

(2.10) Definition (ruler or coordinate system)
Let ℓ be a line in an incidence geometry $\{\mathcal{S}, \mathcal{L}\}$. Assume that there is a distance function d on $\mathcal{S}$. A function $f : \ell \to \mathbb{R}$ is a ruler (or coordinate system) for ℓ if
(i) f is a bijection;
(ii) for each pair of points P and Q on ℓ

$$|f(P) - f(Q)| = d(P, Q). \tag{1}$$

Equation (1) is called the Ruler Equation and $f(P)$ is called the coordinate of P with respect to f.

14. Let $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$ denote the Cartesian Plane and let d denote the Euclidean distance. Define function $f : L_{2,3} \to \mathbb{R}$ on the following way: $f(Q) = f((x, y)) = x\sqrt{5}$, $\forall Q \in L_{2,3}$. Show that f is a ruler for $L_{2,3}$ and find the coordinate of $R(1, 5)$ with respect to f.

15. Let $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$ denote the Poincaré Plane and let d denote the Poincaré distance. Define function $g : {}_4L_9 \to \mathbb{R}$ on the following way: $g(P) = g((x, y)) = \ln\frac{x+5}{y}$, $\forall P \in {}_4L_9$. Show that g is a ruler for ${}_4L_9$ and find the coordinate of $M(5, 2\sqrt{3})$ with respect to g.

16. Let $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$ denote the Cartesian Plane and let d denote the Taxicab distance. Define function $h : L_{-2,3} \to \mathbb{R}$ on the following way: $h(R) = h((x, y)) = 3x$, $\forall R \in L_{-2,3}$. Show that h is a ruler for $L_{-2,3}$ and find the coordinate of $N(1, 1)$ with respect to h.

(2.11) Definition (Ruler Postulate, metric geometry)
An incidence geometry $\{\mathcal{S}, \mathcal{L}\}$ together with a distance function d satisfies the Ruler Postulate if every line $\ell \in \mathcal{L}$ has a ruler. In this case we say $\mathcal{M} = \{\mathcal{S}, \mathcal{L}, d\}$ is a metric geometry.

17. Show that the Cartesian Plane $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$ with the Euclidean distance, d_E, is a metric geometry.

(2.12) Definition (Euclidean Plane)
The Euclidean Plane is the model $\mathcal{E} = \{\mathbb{R}^2, \mathcal{L}_E, d_E\}$.

18. Show that the Poincaré Plane $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$ with the Poincaré distance, d_H, is a metric geometry.

(2.13) Convention (Poincaré Plane)

From now on, the terminology Poincaré Plane and the symbol $\mathcal{H}$ will include the hyperbolic distance d_H:

$$\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H, d_H\}.$$

19. Show that the Cartesian Plane $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$ with the Taxicab distance, d_T, is a metric geometry.

(2.14) Definition (Taxicab Plane)

The model $\mathcal{T} = \{\mathbb{R}^2, \mathcal{L}_E, d_T\}$ will be called the Taxicab Plane.

Lets summarizes the rulers which we have discussed for the three major models of a metric geometry.

Model	Type of line	Standard Ruler or coordinate system for line
Euclidean Plane $\mathcal{E}$	$L_a = \{(a,y) \mid y \in \mathbb{R}\}$ $L_{m,b} = \{(x,y) \in \mathbb{R}^2 \mid y = mx + b\}$	$f(a,y) = y$ $f(x,y) = x\sqrt{1+m^2}$
Poincaré Plane $\mathcal{H}$	${}_aL = \{(a,y) \in \mathbb{H} \mid y > 0\}$ ${}_cL_r = \{(x,y) \in \mathbb{H} \mid (x-c)^2 + y^2 = r^2\}$	$f(a,y) = \ln y$ $f(x,y) = \ln \frac{x-c+r}{y}$
Taxicab Plane $\mathcal{T}$	$L_a = \{(a,y) \mid y \in \mathbb{R}\}$ $L_{m,b} = \{(x,y) \in \mathbb{R}^2 \mid y = mx + b\}$	$f(a,y) = y$ $f(x,y) = (1+\|m\|)x$

Convention. In discussions about one of the three models above, the coordinate of a point with respect to a line ℓ will always mean the coordinate with respect to the standard ruler for that line as given in the above table.

In the next section we will discuss some special rulers for a line. These should not be confused with the standard rulers defined above.

20. In the Euclidean Plane $\mathcal{E} = \{\mathbb{R}^2, \mathcal{L}_E, d_E\}$, (i) find the coordinate of $M(2,3)$ with respect to the line $x = 2$; (ii) find the coordinate of $M(2,3)$ with respect to the line $y = -4x + 11$. (Note that your answers are different.)

21. Find the coordinate of $M(2,3)$ with respect to the line $y = -4x + 11$ for the Taxicab Plane $\mathcal{T} = \{\mathbb{R}^2, \mathcal{L}_E, d_T\}$. (Compare with Problem 20.)

22. Find the coordinates in $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H, d_H\}$ of $M(2,3)$ (i) with respect to the line $(x-1)^2 + y^2 = 10$; (ii) with respect to the line $x = 2$.

23. Find the Poincaré distance between

i. $A(1,2)$ and $B(3,4)$;

ii. $P(2,1)$ and $Q(4,3)$.

24. Find a point P on the line $L_{2,-3}$ in the Euclidean Plane whose coordinate is -2.

25. Find a point P on the line $L_{2,-3}$ in the Taxicab Plane whose coordinate is -2.

26. Find a point P on the line ${}_{-3}L_{\sqrt{7}}$ in the Poincaré Plane whose coordinate is in $\ln 2$.

27. We shall define a new distance d^* on $\mathbb{R}^2$ by using d_E. Specifically:

$$d^*(P,Q) = \begin{cases} d_E(P,Q), & \text{if } d_E(P,Q) \le 1 \\ 1, & \text{if } d_E(P,Q) > 1 \end{cases}$$

(i) Prove that d^* is a distance function. (ii) Find and sketch all points $P \in \mathbb{R}^2$ such that $d^*((0,0),P) \le 2$. (iii) Find all points $P \in \mathbb{R}^2$ such that $d^*((0,0),P) = 2$.

Why do we study metric geometries? It is because many of the concepts in the synthetic approach which must be added are already present in the metric geometry approach. This happens because we can transfer questions about a line ℓ in $\mathcal{L}$, to the real numbers $\mathbb{R}^2$ by using a ruler f. In $\mathbb{R}$ we understand concepts like "between" and so can transfer them back (via f^{-1}) to ℓ. This is the advantage of the metric approach alluded to in the beginning of the section. After we have more background, we will return to the question of a synthetic versus metric approach to geometry.

28. Denote by $\{\mathcal{S},\mathcal{L},d\}$ a metric geometry, let $P \in \mathcal{S}$ denote arbitrary point, $p \in \mathcal{L}$ such that $P \in p$, and let $r \in \mathbb{R}$. Show that on line p there exist at least one point Q such that $d(P,Q) = r$.

29. Let $\{\mathcal{S},\mathcal{L}\}$ be an incidence geometry. Assume that for each line $\ell \in \mathcal{L}$ there exists a bijection $f_\ell : \ell \to \mathbb{R}$. Show that then there is a distance d such that $\{\mathcal{S},\mathcal{L},d\}$ is a metric geometry and each $f_\ell : \ell \to \mathbb{R}$ is a ruler.

30. Let d^* denote distance function on $\mathbb{R}^2$ which is defined on the following way

$$d^*(P,Q) = \begin{cases} d_E(P,Q), & \text{if } d_E(P,Q) \le 1 \\ 1, & \text{if } d_E(P,Q) > 1 \end{cases}$$

Prove that there is no incidence geometry on $\mathbb{R}^2$ such that $\{\mathbb{R}^2,\mathcal{L},d^*\}$ is a metric geometry. (Thus not every distance gives a metric geometry.)

31. If $\{\mathcal{S},\mathcal{L},d\}$ is a metric geometry and $P \in \mathcal{S}$, prove that for any $r > 0$ there is a point in $\mathcal{S}$ at distance r from P.

32. Define the max distance (or supremum distance), d_S, on $\mathbb{R}^2$ by

$$d_S(P,Q) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

where $P(x_1,y_1)$ and $Q(x_2,y_2)$.

(i) Show that d_S is a distance function.

(ii) Show that $\{\mathbb{R}^2,\mathcal{L},d_S\}$ is a metric geometry.

33. In a metric geometry $\{\mathcal{S},\mathcal{L},d\}$ if $P \in \mathcal{S}$ and $r > 0$, then the circle with center P and radius r is $\mathcal{C} = \{Q \in \mathcal{S} \mid d(P,Q) = r\}$. Draw a picture of the circle of radius 1 and center $(0,0)$ in the $\mathbb{R}^2$ for each of the distances d_E, d_T, and d_S.

34. Let $\{\mathcal{S},\mathcal{L},d\}$ be a metric geometry, let $P \in \mathcal{S}$, let $\ell \in \mathcal{L}$ with $P \in \ell$, and let $\mathcal{C} = \{Q \in \mathcal{S} \mid d(P,Q) = r\}$ be a circle with center P. Prove that $\ell \cap \mathcal{C}$ contains exactly two points.

35. Find the circle of radius 1 with center $(0,e)$ in the Poincaré Plane. Hint: As a set this circle "looks" like an ordinary circle. Carefully show this.

36. We may define a distance function for the Riemann Sphere as follows. On a great circle $\mathcal{C}$ we measure the distance $d_R(A,B)$ between two points A and B as the shorter of the lengths of the two arcs of $\mathcal{C}$ joining A to B. (Note $d_R(A,-A) = \pi$.) Prove that d_R is a distance function. Is $\{S^2,\mathcal{L}_R,d_R\}$ a metric geometry?

Solutions: **1.** [d_M is dist fun] **2.** [d_{max} is dist fun] **3.** [d is dist. fun.] **7.** [$P(\frac{t}{\sqrt{1+m^2}}, \frac{mt}{\sqrt{1+m^2}} + b)$, $P \in L_{m,b}$, $f(P) = t$] **8.** [$g^{-1}(t) = (a, e^t)$] **9.** [$f(P) = f(Q) \Rightarrow P = Q$] **11.** [$\ln(x - c + r) - \ln(y) = t$, $e^t = \frac{x-c+r}{y}$, $e^{-t} = -\frac{x-c-r}{y}$, $y = r\operatorname{sech}(t)$, $x - c = r\tanh(t)$]